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Eikonal amplitude in the gravireggeon model at superplanckian energies

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Abstract. The gravity effects in high-energy scattering, described by a four-dimensional eikonal amplitude related to gravireggeons induced by compact extra dimensions are studied. It is demonstrated that the real part of the eikonal (with a massless mode subtracted) dominates its imaginary part at both small and large impact parameters, in contrast with the usual case of hadronic high-energy behavior. The real part of the scattering amplitude exhibits an exponential falloff at large momentum transfer, similar to that of the imaginary part of the amplitude.

1 Introduction

In our previous paper [1], we considered the model with compact extra spatial dimensions [2] and calculated the contribution of Kaluza–Klein (KK) gravireggeons into the inelastic cross-section of high energy scattering of four-dimensional SM particles. In particular, an expression for the imaginary part of the eikonal has been derived. The results were applied to cosmic neutrino gravitational interaction with atmospheric nucleons [1].

In the present paper, we study quantum gravity effects related to the extra dimensions in the *real part of the eikonal*. As in [1], the SM particles are confined on a four-dimensional brane, while the gravity lives in all $D = d + 4$ dimensions. The extra dimensions are compactified, with a radius R_c. Thus, a fundamental mass scale, M_D, is related to the Planck scale by a relation $M_{Pl}^2 = M_D^{d+2}\,(2\pi R_c)^d$ [2].

In the next section, we consider a case of one extra dimension. The generalization to more than one extra dimension is given in Sect. 3. The conclusions and discussions are given in the last section. Some technical details of our calculations are collected in the Appendices.

2 One extra dimension ($d = 1$)

For the sake of simplicity and for pedagogical reasons, we will consider first one extra dimension. The general case ($d > 2$) will be analyzed in the next section. In the gravireggeon model, eikonal is given by the sum of reggeized KK gravitons in the t-channel [1]:

$$\chi(s,b) = \frac{1}{8\pi s} \int\limits_{-\infty}^{0} dt\, J_0(b\sqrt{-t}) \sum_{n=-\infty}^{\infty} A^B(s,t,n), \qquad (1)$$

where $\sqrt{s}$ is an invariant energy and the Born amplitude is of the form

$$A^B(s,t,n) = G_N \left[i - \cot\frac{\pi}{2}\alpha_n(t) \right] \alpha'_g\, \beta_n^2(t) \left(\frac{s}{s_0} \right)^{\alpha_n(t)}. \qquad (2)$$

Here, n is a KK-number. The value $n = 0$ corresponds to usual massless graviton.

Both massless graviton and its KK massive excitations lie on linear Regge trajectories:

$$\alpha(t_D) = \alpha(0) + \alpha'_g\, t_D, \qquad (3)$$

where t_D denotes D-dimensional momentum transfer. Because the extra dimension is compact with the radius R_c, we come to splitting of the Regge trajectory (3) into a leading vacuum trajectory,

$$\alpha_0(t) \equiv \alpha_{grav}(t) = 2 + \alpha'_g t, \qquad (4)$$

and infinite sequence of secondary, KK-charged gravireggeons [3]:

$$\alpha_n(t) = 2 - \frac{\alpha'_g}{R_c^2}\, n^2 + \alpha'_g t, \quad n \geqslant 1. \qquad (5)$$

The string theory implies that the slope of the gravireggeon trajectory is universal for all s, and $\alpha'_g = 1/M_s^2$, where M_s is a string scale.

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In [1], the imaginary part of the eikonal (1) has been calculated. In the present paper, we consider the real part of the eikonal. From (1), (2) and (5), we obtain ($q_\perp^2 = -t$):

$$\mathrm{Re}\,\chi(s,b) = G_N s \frac{\alpha_g'}{8} \int_0^\infty q_\perp dq_\perp \, J_0(q_\perp b)\, e^{-q_\perp^2 R_g^2(s)} \times \sum_{n=-\infty}^{\infty} \cot\Big[\frac{\pi\alpha_g'}{2}\big(-t + \frac{n^2}{R_c^2}\big)\Big] e^{-n^2 R_g^2(s)/R_c^2}, \quad (6)$$

where

$$R_g(s) = \sqrt{\alpha_g' \ln(s/s_0)} \quad (7)$$

is a gravitational slope (dynamic radius). Formally, there exist poles in the sum in (6) at negative values of $\alpha_n(t)$. It is demonstrated in Appendix A that these tachyon poles are fictitious singularities, and, thus, they will not be taken into account in our calculations.

In what follows, we will assume that t lies in the physical region, $t < 0$, and

$$\alpha_g' \, |t| \ll 1. \quad (8)$$

It is equivalent to $|t| \ll M_s^2$, where the string scale M_s is of order 1 TeV. The sum in (6) is effectively cut from above, $n \lesssim n_{\max} = R_c/R_g(s)$. It means that $\alpha_g' n^2/R_c^2 \lesssim [\ln(s/s_0)]^{-1} \ll 1$ in (6).

Let us define $\mathrm{Re}\,\tilde\chi(s,b)$ to be the real part of the eikonal with a pole term (corresponding to $n = 0$ in (6)) subtracted. With the abovementioned, it can be written as follows:[1]

$$\mathrm{Re}\,\tilde\chi(s,b) = G_N R_c^2\, s \frac{1}{2\pi} \int_0^\infty q_\perp dq_\perp \, J_0(q_\perp b)\, e^{-q_\perp^2 R_g^2(s)} \times \sum_{n=1}^{\infty} \frac{1}{n^2 + R_c^2|t|}\, e^{-n^2 R_g^2(s)/R_c^2}. \quad (9)$$

One can see that $\varepsilon(s) = R_g(s)/R_c \ll 1$ even at ultra-high (cosmic) energies, s. Indeed, a magnitude of $\varepsilon(s)$ is defined by the ratio $\sim M_c/M_s$, with a compactification mass scale, $M_c = R_c^{-1}$, varying from 10^{-3} eV for $d = 2$ to 10 MeV for $d = 6$. So, $\varepsilon(s)$ is taken to be a small parameter everywhere in our calculations.

Let us consider two distinct regions of the momentum transfer $|t|$. For $0 \leqslant |t| \ll R_c^{-2}$, the leading term looks like

$$I_1 = \sum_{n=1}^{\infty} \frac{1}{n^2 + R_c^2|t|}\, e^{-n^2 R_g^2(s)/R_c^2}\Big|_{|t|R_c^2 \ll 1} \simeq \frac{\pi^2}{6} - \frac{\pi^4}{90} R_c^2\, |t| + \mathrm{O}(\varepsilon(s)). \quad (10)$$

At large $|t|$ ($R_c^{-2} \ll |t| < \infty$), we will consider two subregions. If the momentum transfer runs the subregion $R_c^{-2} \ll |t| \ll R_g^{-2}(s)$, then

$$I_1 = \sum_{n=1}^{\infty} \frac{1}{n^2 + R_c^2|t|}\, e^{-n^2 R_g^2(s)/R_c^2}\Big|_{\substack{|t|R_c^2 \gg 1 \\ |t|R_g^2(s) \ll 1}} \simeq \frac{\pi}{2R_c\sqrt{|t|}} - \frac{1}{R_c^2|t|} + \mathrm{O}(\varepsilon(s)). \quad (11)$$

Note, the leading terms in (10) and (11) match at $R_c\sqrt{|t|} = a_1 = 3/\pi$. At very large values of $|t|$, namely, for $R_g^{-2}(s) \lesssim |t| < \infty$, the sum in (11) has the asymptotics

$$I_1\Big|_{|t|R_g^2(s) \gg 1} \simeq \frac{\pi}{2R_c R_g(s)|t|}. \quad (12)$$

It can be shown that a contribution from the region $R_g^{-2}(s) \lesssim |t| < \infty$ is suppressed as compared with the region $R_c^{-2} \lesssim |t| < \infty$ by the factor $\sim \varepsilon(s)$ (for large impact parameter, which we are interested in). Thus, we can write (using table integrals from [5]):

$$\mathrm{Re}\,\tilde\chi(s,b) \simeq G_N R_c^2 s \Big[\frac{\pi}{12} \int_0^{a_1 R_c^{-1}} q_\perp dq_\perp \, J_0(q_\perp b)\, e^{-q_\perp^2 R_g^2(s)} + \frac{1}{4R_c} \int_{a_1 R_c^{-1}}^\infty dq_\perp \, J_0(q_\perp b)\, e^{-q_\perp^2 R_g^2(s)}\Big]$$
$$\simeq \frac{G_N R_c s}{4} \left\{ \frac{1}{b} J_1\left(\frac{a_1 b}{R_c}\right) + \frac{a_1}{R_c} \times \left[\frac{\sqrt{\pi} R_c}{2 a_1 R_g(s)}\, \Phi\left(\frac{1}{2},\, 1;\, -\frac{b^2}{4R_g^2(s)}\right) - {}_1F_2\left(\frac{1}{2};\, 1,\, \frac{3}{2};\, -\frac{a_1^2 b^2}{4R_c^2}\right) \right] \right\}, \quad (13)$$

where $\Phi(a;b;z)$ is the confluent hypergeometric function,[2] and ${}_1F_2(a;b,c;z)$ is the generalized hypergeometric function [6]. For $b \gg R_c^2/R_g(s) \gg R_c$, we get the following asymptotics [6]:

$$\frac{\sqrt{\pi} R_c}{2 a_1 R_g(s)}\, \Phi\left(\frac{1}{2},\, 1;\, -\frac{b^2}{4R_g^2(s)}\right)\Big|_{b\varepsilon(s) \gg R_c} \simeq \frac{R_c}{a_1 b}\left[1 + \mathrm{O}\left(\frac{R_g^2(s)}{b^2}\right)\right], \quad (14)$$

and[3]

$${}_1F_2\left(\frac{1}{2};\, 1,\, \frac{3}{2};\, -\frac{a_1^2 b^2}{4R_c^2}\right)\Big|_{b \gg R_c} \simeq \frac{R_c}{a_1 b}\left[1 + J_1\left(\frac{a_1 b}{R_c}\right) + \frac{R_c}{a_1 b} J_2\left(\frac{a_1 b}{R_c}\right)\right]. \quad (15)$$

Here, $J_1(z)$ and $J_2(z)$ are the Bessel functions. As a result, we obtain from (13) and (14)–(15):

$$\mathrm{Re}\tilde\chi(s,b)\Big|_{b \gg R_c} \simeq -G_N s \frac{\pi}{12}\left(\frac{R_c}{b}\right)^2 J_2\left(\frac{a_1 b}{R_c}\right), \quad (16)$$

[1] Taking into account that, effectively, $|\alpha_n(t) - 2| \ll 1$.

[2] The confluent hypergeometric function $\Phi(1/2,\, 1;\, z)$ can be related to the modified Bessel function $I_0(z/2)$.

[3] The generalized hypergeometric function ${}_1F_2(1/2;\, 1,\, 3/2;\, -z)$ can be defined by the integral of the Bessel function $J_0(z)$. As a result, its asymptotics have oscillations.

where the constant a_1 is defined after formula (11).

At zero impact parameters, one has

$$\mathrm{Re}\,\tilde{\chi}(s, b=0) \sim G_N s \frac{R_c}{R_g(s)}. \tag{17}$$

On the other hand, the imaginary part of the eikonal was found to be (for $d = 1$) [1]

$$\mathrm{Im}\,\chi(s,b) = \sqrt{\pi}\, G_N s \frac{R_c}{R_g(s)} \left[\ln(s/s_0)\right]^{-1} \exp[-b^2/4R_g^2(s)]. \tag{18}$$

So the real part of the eikonal dominates the imaginary part at zero impact parameter,[4] $\mathrm{Re}\,\tilde{\chi}(s,0)/\mathrm{Im}\,\tilde{\chi}(s,0) \sim \ln s$, and it has a powerlike behaviour (with oscillations) at $b \to \infty$ (16), while the imaginary part decreases exponentially at large b.

3 More than two extra dimensions ($d > 2$)

The expression for the real part of the eikonal (9) is easily generalized for $d > 2$:

$$\mathrm{Re}\,\tilde{\chi}(s,b) = G_N R_c^2 s \frac{1}{2\pi} \int_0^\infty q_\perp dq_\perp\, J_0(q_\perp b)\, e^{-q_\perp^2 R_g^2(s)} \times \sum_{n=1}^{\infty} \frac{1}{n^2 + R_c^2|t|}\, e^{-n^2\varepsilon^2(s)} \sum_{n_1^2+n_2^2+\cdots n_{d-1}^2 \leqslant n^2}, \tag{19}$$

where the notation $n^2 = \sum_i^d n_i^2$ is introduced. The main contribution to the sum in the right-hand side of (19) comes from the region $n^2 \sim (d-2)/\varepsilon^2(s) \gg 1$. Thus, to estimate the sum in $(n_1, n_2, \ldots n_{d-1})$ analytically, we can replace the sum by the integral

$$\begin{aligned} I_d &= \sum_{n=1}^{\infty} \frac{1}{n^2 + R_c^2|t|}\, e^{-n^2\varepsilon^2(s)} \sum_{n_1^2+n_2^2+\cdots n_{d-1}^2 \leqslant n^2} \\ &\to \sum_{n=1}^{\infty} \frac{1}{n^2 + R_c^2|t|}\, e^{-n^2\varepsilon^2(s)} \int \cdots \int_{\mathbf{x}^2 \leqslant n^2} d\mathbf{x} \\ &= \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \sum_{n=1}^{\infty} \frac{n^{d-1}}{n^2 + R_c^2|t|}\, e^{-n^2\varepsilon^2(s)}. \end{aligned} \tag{20}$$

As in the previous section, we consider two regions of the momentum transfer. At $|t| \ll R_c^{-2}$, we obtain (up to corrections $\mathrm{O}(R_c^2|t|)$ and $\mathrm{O}(\varepsilon^2(s))$):

$$\begin{aligned} I_d &\simeq \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \times \sum_{n=1}^{\infty} \frac{1}{n^2 + R_c^2|t|}\left(n^{d-1} - 1\right) e^{-n^2\varepsilon^2(s)} \\ &\to \frac{\pi}{6} + \frac{\pi^{(d-1)/2}}{2\Gamma\left(\frac{d+1}{2}\right)} \int_1^\infty dz \left(z^{d/2-2} - z^{-3/2}\right) e^{-z\varepsilon^2(s)} \\ &\simeq \frac{\pi}{6} + \frac{\pi^{(d-1)/2}}{2\Gamma\left(\frac{d+1}{2}\right)} \left[\Psi\left(1, \frac{d}{2}; \varepsilon^2(s)\right) - \Psi\left(1, \frac{1}{2}; \varepsilon^2(s)\right)\right], \end{aligned} \tag{21}$$

where $\Psi(a,b;z)$ is the confluent hypergeometric function,[5] and we have replaced the sum in n by the integral.[6] For $d > 2$, (21) results in

$$I_d\Big|_{|t|R_c^2 \ll 1} \simeq \frac{\pi^{(d-1)/2}\Gamma\left(\frac{d}{2}-1\right)}{2\Gamma\left(\frac{d+1}{2}\right)} \left(\frac{1}{\varepsilon(s)}\right)^{d-2}, \tag{22}$$

neglecting insignificant terms $\mathrm{O}(\varepsilon(s))$. Starting from (21), we come to the asymptotics $I_1 \simeq \pi^2/6$, in accordance with (10).

Now let us consider the region $R_c^{-2} \ll |t| < \infty$. Then the quantity I_d can be cast in the form:

$$\begin{aligned} I_d &\simeq \frac{\pi^{(d-1)/2}}{2\Gamma\left(\frac{d+1}{2}\right)} \int_1^\infty dz \frac{z^{d/2-1}}{z + R_c^2|t|}\, e^{-z\varepsilon^2(s)} \\ &\simeq \frac{\pi^{(d-1)/2}}{2\Gamma\left(\frac{d+1}{2}\right)} \left[\Gamma\left(\frac{d}{2}\right)(R_c^2|t|)^{d/2-1}\, \Psi\left(\frac{d}{2}, \frac{d}{2}; R_g^2(s)|t|\right) - \frac{2}{d}\frac{1}{R_c^2|t|}\right]. \end{aligned} \tag{23}$$

For $d > 2$, we should consider separately two subregions. Namely, if the momentum transfer is bounded by inequalities $R_c^{-2} \ll |t| \ll R_g^{-2}(s)$, (23) results in previously obtained asymptotics (22). At very large values of $|t|$, such as $R_g^{-2}(s) \ll |t| < \infty$, one gets from (23)

$$I_d\Big|_{|t|R_c^2 \gg 1} \simeq \frac{\pi^{(d-1)/2}\Gamma\left(\frac{d}{2}\right)}{2\Gamma\left(\frac{d+1}{2}\right)} \left(\frac{1}{\varepsilon(s)}\right)^{d-2} \frac{1}{R_g^2(s)|t|}. \tag{24}$$

For $d = 1$, the correct values of I_1 (11), (12) are reproduced. The asymptotics (22) and (24) match at $|t| = a^2 R_g^{-2}(s)$, where

$$a^2 = (d-2)/2. \tag{25}$$

Thus, we get the following expression for the eikonal:

$$\mathrm{Re}\,\tilde{\chi}(s,b) \simeq G_N s \left[\frac{R_c}{R_g(s)}\right]^d \frac{\pi^{(d-3)/2}\Gamma\left(\frac{d}{2}\right)}{4\Gamma\left(\frac{d+1}{2}\right)} \times \left[\frac{R_g^2(s)}{a^2} \int_0^{aR_g^{-1}(s)} q_\perp dq_\perp\, J_0(q_\perp b)\, e^{-q_\perp^2 R_g^2(s)} + \int_{aR_g^{-1}(s)}^\infty \frac{dq_\perp}{q_\perp} J_0(q_\perp b)\, e^{-q_\perp^2 R_g^2(s)}\right]$$

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[4] We remind the reader that the singular term was subtracted in $\mathrm{Re}\,\tilde{\chi}$.

[5] The functions $\Psi(a,b;z)$ and the abovementioned function $\Phi(a,b;z)$ are different solutions of the confluent hypergeometric equation.

[6] Note, for $d = 1$, this is justified only if $|t|R_c^2 \gg 1$.

$$\equiv G_N s \left[\frac{R_c}{R_g(s)}\right]^d \frac{\pi^{(d-3)/2}\Gamma\left(\frac{d}{2}\right)}{4\Gamma\left(\frac{d+1}{2}\right)} \times \left[I_<\left(\frac{ab}{R_g(s)}\right) + I_>\left(\frac{ab}{R_g(s)}\right)\right]. \quad (26)$$

At small impact parameters, we get immediately, from (26),

$$\mathrm{Re}\,\tilde{\chi}(s,b)\Big|_{b \ll R_g(s)} = C(d)\, G_D s\, \alpha_g'^{-d/2}\left[\ln\left(\frac{s}{s_0}\right)\right]^{-d/2} + \mathrm{O}\left(\frac{b^2}{R_g^2(s)}\right), \quad (27)$$

where $C(d)$ is a constant depending on the number of the extra dimensions, the explicit form of which can be obtained from (26). The asymptotics of the real part of the eikonal at large b is more complicated to analyze. For $b \gg R_g(s)$, it is calculated in Appendix B, and the leading term looks like

$$\mathrm{Re}\tilde{\chi}(s,b)\Big|_{b \gg R_g(s)} \simeq -G_D s\, \alpha_g'^{-d/2}\, \frac{e^{-a^2}\,\Gamma\left(\frac{d}{2}\right)}{\pi^{(d+3)/2}2^{d+1}a^2\Gamma\left(\frac{d+1}{2}\right)} \times \left[\ln\left(\frac{s}{s_0}\right)\right]^{-d/2}\left(\frac{R_g(s)}{b}\right)^2 J_2\left(\frac{ab}{R_g(s)}\right). \quad (28)$$

The expression for the imaginary part of the eikonal for $d \geqslant 1$ was calculated in [1]:

$$\mathrm{Im}\,\chi(s,b) = \frac{G_D s\, \alpha_g'^{-d/2}}{\pi^{d/2-1}}\left[\ln\left(\frac{s}{s_0}\right)\right]^{-(1+d/2)} \times \exp[-b^2/4R_g^2(s)]. \quad (29)$$

As one can see from (27) and (29),

$$\frac{\mathrm{Re}\,\tilde{\chi}(s,0)}{\mathrm{Im}\,\chi(s,0)} \sim \ln s. \quad (30)$$

Let us stress, we study the case when *colliding particles are confined on the four-dimensional brane*, with gravity living in all D dimensions. We see that the real part of the eikonal in b-space has a powerlike behavior in b with oscillations, while the imaginary part decreases exponentially at $b \gg 2\alpha_g' \ln s$. Both depend on the Regge slope α_g' via the gravitational radius $R_g(s)$ (7).[7]

It is interesting to compare our results with the well-known asymptotic behavior of the eikonal function derived in the framework of the string theory for the scattering of *D-dimensional fields* (gravitons) in a flat space-time [10]:

$$\chi_D(s,b)\Big|_{b^2 \gg \alpha' \ln s} \simeq \left(\frac{b_c}{b}\right)^d + i\pi^2\frac{G_N^D s\,\alpha'^{-d/2}}{(\pi \ln s)^{1+d/2}}\exp\Big(-\frac{b^2}{4\alpha' \ln s}\Big), \quad (31)$$

[7] It does not take place, if colliding particles live in D dimensions [9].

where $b_c = [2\pi^{-d/2}\Gamma(d/2)G_N^D s]^{1/d}$, G_N^D being the Newton constant in D flat dimensions. Note, the real part of $\chi_D(s,b)$ exhibits power-law falloff, which does not depend on the string tension $\alpha' \equiv \alpha_g'$.

As one can see, the real part of the eikonal (28) decreases as a *fixed (d-independent) power of* b at $b \to \infty$, contrary to asymptotics (31). The scales in the real parts (associated with the impact parameter b) are also different: dynamic radius $R_g(s)$ in our case, and $b_c \sim (G_N^D s)^{1/d}$ in $\chi_D(s,b)$ (31). It is not strange because we solve a different problem than that considered in [10]. Namely, we are interested in the scattering of four-dimensional particles, while the authors in [10] studied the collisions of the gravitons, which are allowed to propagate in the bulk.

The more amazing thing is that the *imaginary parts* of $\chi(s,b)$ and $\chi_D(s,b)$ *coincide* at $b \gg \alpha_g' \ln s$. It becomes clear, if one takes into account the definition of the gravitational radius, $R_g(s)$ (7). We think it is related to the fact that the exchange quanta are D-dimensional fields in both cases.

Because of the inequality $R_g(s) \ll R_c$, our formulae contain the compactification radius R_c only via D-dimensional coupling $G_D = M_D^{-(2+d)} = G_N(2\pi R_c)^d$. However, at extremely high energies, when the dynamic radius $R_g(s)$ becomes comparable with (or larger than) R_c, the eikonal profile in impact parameter space should feel the size of the compact dimensions, R_c [1].

In this connection, let us mention the SM in a *D-dimensional space-time* with compact extra dimensions, but *without gravity* [11]. In such a case, the dynamic radius, $R(s)$, is proportional to $\ln(s/s_0)/\sqrt{t_0}$, where t_0 denotes the nearest (nonzero) singularity in the t-channel (for instance, $t_0 = m_\pi^2$, if only strong interactions are taken into account).

The expression for four-dimensional eikonal amplitude (in the presence of d compact extra dimensions) looks like [1]

$$A(s,t) = 2is\int d^2b\, e^{iq_\perp b}\left[1 - e^{i\chi(s,b)}\right]. \quad (32)$$

At not extreme energies, namely, for $\sqrt{s} \lesssim M_D \sim M_s$, we have inequalities $\mathrm{Re}\,\tilde{\chi}(s,b), \mathrm{Im}\,\chi(s,b) \ll 1$, and (32) is given by

$$\tilde{A}(s,t) \simeq 4\pi s\int_0^\infty db\, b\, J_0(q_\perp b)\,[\mathrm{Re}\,\tilde{\chi}(s,b) + i\,\mathrm{Im}\,\chi(s,b)] = \mathrm{Re}\tilde{A}(s,t) + i\,\mathrm{Im}\,A(s,t). \quad (33)$$

The imaginary part of the scattering amplitude exhibits exponential falloff at large $|t|$:

$$\mathrm{Im}\,A(s,t) = \frac{8\,G_D s^2\,\alpha_g'^{1-d/2}}{\pi^{d/2-2}}\left[\ln\left(\frac{s}{s_0}\right)\right]^{-d/2} \times \exp\left(t\alpha_g' \ln(s/s_0)\right). \quad (34)$$

As for the real part of the amplitude, we obtain the following behaviour (see Appendix C for details):

$$\mathrm{Re}\tilde{A}(s,t) = G_D s^2 \, \alpha_g'^{-d/2} \, \frac{\Gamma\left(\frac{d}{2}\right)}{2^d \pi^{(d+1)/2} \Gamma\left(\frac{d+1}{2}\right)} \left[\ln\left(\frac{s}{s_0}\right)\right]^{-d/2} \times \begin{cases} \dfrac{\alpha_g' \ln(s/s_0)}{a^2} \left[2 + \dfrac{\alpha_g' \ln(s/s_0)\, t}{a^2}\right] \\ + \dfrac{1}{t}\left[1 - \exp\left(t\alpha_g' \ln(s/s_0)\right)\right] \\ \left(0 < \alpha_g' \, |t| < a^2/\ln(s/s_0)\right) \\ \dfrac{1}{-t} \exp\left(t\alpha_g' \ln(s/s_0)\right) \\ \left(\alpha_g' \, |t| \geqslant a^2/\ln(s/s_0)\right) \end{cases} \quad (35)$$

Note that $\mathrm{Im}\, A(s,t) \ll \mathrm{Re}\, \tilde{A}(s,t)$ in the kinematical region (8), in particular,

$$\left. \frac{\mathrm{Re}\, \tilde{A}(s,t)}{\mathrm{Im}\, A(s,t)} \right|_{t=0} \sim \ln s, \qquad \left. \frac{\mathrm{Re}\, \tilde{A}(s,t)}{\mathrm{Im}\, A(s,t)} \right|_{1 \gg \alpha_g' \, |t| \gg (\ln s)^{-1}} \sim \frac{1}{\alpha_g' \, |t|}. \quad (36)$$

The asymptotics of the amplitude at large $|t|$ (35) is quite different from the behavior of the eikonal amplitude in both the string theory [10] and in the model with Regge exchanges in *D flat dimensions* [9]:

$$A(s,t)\Big|_{|t| \gg b_c^{-2}} \sim G_N^{D} s^2 \, \alpha_g'^{(1-d)/2} |t|^{-(d+2)^2/4(d+1)} \, e^{i\phi_D(t)}, \quad (37)$$

where $\phi_D(t) \sim |t|^{d/2(d+1)}$ and b_c is define after formula (31). Formula (35) is also different from the asymptotic behavior of $A(s,t)$ in the model with *compact extra dimensions*, when nonreggeized KK graviton exchanges are summed up [12]:

$$A(s,t)\Big|_{|t| R_c^2 \gg 1} \sim G_D s^2 \, \alpha_g'^{(1-d)/2} |t|^{-(d+2)/2(d+1)} \, e^{i\phi_D(t)}. \quad (38)$$

It is worth noting that (38) decreases as a power of $|t|$ (the latter being larger than -1 for $d \geqslant 1$), in spite of the fact that both amplitudes describe the scattering of fields trapped *on the brane*. This can be understood as follows. For nonreggeized exchanges [12], the sum in KK numbers $(n_1, n_2, \dots n_d)$ contains no suppression factor $\exp[-n^2 R_g^2(s)/R_c^2]$, contrary to our approach with the gravireggeon exchanges (19). The sum diverges and needs a definition for $d > 1$. Usually, the sum is replaced by a d-dimensional integral, which is calculated by using dimensional regularization. This procedure leads to the power-like falloff of the eikonal with d-depending power, similar to the case when colliding fields are not confined to the brane, but can propagate in the extra dimensions [10]. Moreover, the eikonal is pure real in this scheme [12].

In order to get one or another regime (D-dimensional or four-dimensional one), one has to compare the effective gravitational radius $R_g(s)$ with the compactification radius R_c, but not the impact parameter b with $R_g(s)$ or R_c. To illustrate it, let us analyze the imaginary part of the eikonal, which was calculated analytically in our previous paper [1].

For the sake of simplicity (but without loss of generality), we consider the case $d = 1$. The imaginary part of the eikonal can be represented in the following general form [1]:

$$\mathrm{Im}\, \chi(s,b) = G_N s \, \frac{\alpha_g'}{R_g^2(s)} \exp\Big[- b^2/4R_g^2(s)\Big] \, \theta_3(0,q), \quad (39)$$

where $q = \exp(-R_g^2(s)/R_c^2)$ and θ_3 is one of Jacobi θ-functions [13]:

$$\theta_3 = 1 + 2\sum_{n=1}^{\infty} \exp\left[-n^2 \, R_g^2(s)/R_c^2\right]. \quad (40)$$

Thus, $\theta_3 \simeq 1$ at $R_g(s) \gg R_c$, and we reproduce a four-dimensional result in this limit (only zero mode contributes).

On the other hand, we can use unimodular transformation of θ_3 and get [13]

$$\theta_3 = \frac{\sqrt{\pi}\, R_c}{R_g(s)} \left\{ 1 + 2\sum_{k=1}^{\infty} \exp\left[-(2\pi k)^2 \, R_c^2/R_g^2(s)\right] \right\}. \quad (41)$$

At $R_g(s) \ll R_c$, we obtain from (41) that $\theta_3 \simeq \sqrt{\pi}\,(R_c/R_g(s))$, and we come to a five-dimensional expression (18) (remember that $G_N R_c \sim G_5$).

Note that the b-dependence of $\mathrm{Im}\, \chi(s,b)$ is the same in both limits, that is, it does not depend on the ratio $R_g(s)/R_c$.

4 Conclusions

In the framework of the model with d extra compact dimensions, we have calculated the quantum gravity effects related to the gravireggeon exchanges in t-channel. For the scattering of the SM fields living on the four-dimensional brane, the real part of the eikonal (with the massless mode subtracted) is estimated. It is shown that it decreases as a power of b (with oscillations) at large values of the impact parameter b. This power does not depend on the number of the extra dimensions d, contrary to the case when the colliding fields are allowed to propagate in the bulk [10]. The scale, associated with the impact parameter b, is $\alpha_g'^{-1}$, while in the D-dimensional flat space-time, the corresponding scale is defined by $(G_N^{D} s)^{1/d}$, where G_N^{D} is the Newton constant in D dimensions.

Let us emphasise that we have calculated the eikonal and the scattering amplitude for the case when the colliding particles are confined to the four-dimensional brane, while the classical results of [10] correspond to the case when the colliding fields can propagate in the extra dimensions.

The calculations complete our results on the imaginary part of the eikonal obtained previously. In particular, it was shown that the imaginary parts of the eikonal are the same for the case when colliding particles are confined to the brane and when they propagate freely in extra dimensions. In the present paper, we have also calculated the eikonal amplitude and have shown that both the real part of the amplitude and its imaginary part decreases exponentially at large momentum transfer.

The real part of the amplitude dominates the imaginary part at zero momentum transfer, in contrast with high-energy behaviour of hadronic amplitudes (see, for instance, [14]). Note, however, that this result was obtained in the region $\ln s \ll R_c^2/\alpha_g'$. At asymptotical s, the inequality $|\mathrm{Re}\, A(s,0)|/|\mathrm{Im}\, A(s,0)| < \mathrm{const}$ will be reproduced, provided the massless mode is discarded in the eikonal.

Appendix A

The Sommerfeld–Watson transformation results in the following expression for a contribution of the Regge trajectory $\alpha(t)$ to the amplitude [4]:

$$A(s,t)\Big|_{pole} = -16\pi^2[2\alpha(t)+1]\beta(t) \times \left[\frac{1+\xi\exp(-i\pi\alpha(t))}{\sin\pi\alpha(t)}\, P_{\alpha(t)}(-z_t(s,t)) - \xi\frac{2}{\pi}\, Q_{\alpha(t)}(-z_t(s,t)) \right]. \quad \text{(A.1)}$$

Here, ξ is a signature of the trajectory, $z_t(s,t)$ is a cosine of a scattering angle in the t-channel, $\beta(t)$ is a residue of the Regge pole in a partial amplitude:

$$A_l^\xi(t)\Big|_{l\to\alpha} \simeq \frac{\beta(t)}{l-\alpha(t)}. \quad \text{(A.2)}$$

We have omitted a background integral in (A.1), which is nonleading in the high-energy limit (that is, at $-z_t(s,t) \gg 1$). Note that the second term in the right-hand side of (A.1) is usually discarded because it is also negligible at $-z_t(s,t) \gg 1$. Nevertheless, it becomes important if we look for possible nonphysical singularities.

For even signature ($\xi = +1$), one gets the real part of the amplitude in the form:[8]

$$\mathrm{Re}A(s,t)\Big|_{pole} = -16\pi^2(2\alpha+1)\,\beta \times \left[\frac{1+\cos\pi\alpha}{\sin\pi\alpha}\, P_\alpha(-z) - \frac{2}{\pi}\, Q_\alpha(-z) \right], \quad \text{(A.3)}$$

where simplified notations $\alpha \equiv \alpha(t)$, $\beta \equiv \beta(t)$ and $z \equiv z_t(s,t)$ are introduced. In order to analyse singularities of the amplitude in α, it is convenient to represent the expression in the right-hand side of (A.3) via hypergeometric functions [6]:

[8] Note that $\mathrm{Im}P_l(z) = \mathrm{Im}Q_l(z) = 0$ at $z > 1$, for any real l [6].

$$\frac{1+\cos\pi\alpha}{\sin\pi\alpha}\, P_\alpha(-z) - \frac{2}{\pi}\, Q_\alpha(-z) = -\frac{\pi^{-1/2}}{\cos\pi\alpha} \Bigg[(-2z)^\alpha\, \frac{(1+\cos\pi\alpha)\,\Gamma(-\alpha)}{\Gamma(-\alpha+1/2)} \times {}_2F_1\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+\frac{1}{2}; -\frac{\alpha}{2}+1; \frac{1}{z^2}\right) - (-2z)^{-\alpha-1}\, \frac{(1-\cos\pi\alpha)\,\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)} \times {}_2F_1\left(\frac{\alpha}{2}+\frac{1}{2}, \frac{\alpha}{2}+1; \frac{\alpha}{2}+\frac{3}{2}; \frac{1}{z^2}\right) \Bigg]. \quad \text{(A.4)}$$

This expression is symmetric under replacement $\alpha \to -\alpha-1$. Note that the ratio ${}_2F_1(a,\, b;\, c; z)/\Gamma(c)$ has neither singularities nor zeros in c.

It follows from (A.4) that the amplitude has two sets of simple poles: physical singularities at $\alpha(t) = 2m$, $m = 0, 1 \ldots$, and tachyon poles at $\alpha(t) = -(2n+1)$, $n = 0, 1 \ldots$. If the term $(2/\pi)\, Q_\alpha(-z)$ is disregarded in (A.3), the tachyon poles are shifted to the points $\alpha(t) = -2n$. Notice that there are no poles in the right-hand side of (A.4) at half-integer α because an expression in square brackets tends to zero and cancels zeros of a function $\cos\pi\alpha$ at these points.

Let us demonstrate that the tachyon poles are fictitious ones. In order to do this, we will consider the Mandelstam-Sommerfeld–Watson transformation, which is based on using the Legendre function of the second kind [4]. By disregarding all the terms but the pole contribution and the sum in positive angular momenta, we get

$$A'(s,t) = 16\pi\,[1+\xi\exp(i\pi\alpha(t))] \left[(2\alpha(t)+1)\beta(t)\frac{Q_{-\alpha(t)-1}(-z_t(s,t))}{\cos\pi\alpha(t)} - \frac{1}{\pi}\sum_{l=1}^{\infty}(-1)^{l-1}(2l)\, A_{l-1/2}^\xi(t)\, Q_{l-1/2}(-z_t(s,t)) \right]. \quad \text{(A.5)}$$

In particular, the contribution of the Regge trajectory (with $\xi = +1$) into the real part of the amplitude,

$$\mathrm{Re}A'(s,t)\Big|_{pole} = 16\pi(2\alpha+1)\,\beta\,\frac{1+\cos\pi\alpha}{\cos\pi\alpha}\, Q_{-\alpha-1}(-z), \quad \text{(A.6)}$$

reveals *the same* physical singularities (at $\alpha(t) = 2m$, $m = 0, 1 \ldots$), as can be easily seen from the relation [6]

$$Q_{-\alpha-1}(-z) = \pi^{1/2}\,(-2z)^\alpha\,\frac{\Gamma(-\alpha)}{\Gamma(-\alpha+1/2)} \times {}_2F_1\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+\frac{1}{2}; -\frac{\alpha}{2}+1; \frac{1}{z^2}\right). \quad \text{(A.7)}$$

The zeros of the function $\cos\pi\alpha$ in (A.5) can result in singularities of $\mathrm{Re}A'(s,t)$ at half-integer α. However, the poles of $(\cos\pi\alpha)^{-1}$ at $\alpha(t) = n+1/2$, with $n = 0, 1 \ldots$, and corresponding poles of the partial amplitudes in the sum in (A.5) cancel out. To see this, one should use the formula

$$Q_{l-1/2}(z) = Q_{-l-1/2}(z), \quad \text{(A.8)}$$

valid for any integer l [6]. Only tachyon poles, $\alpha(t) = -(n+1/2)$, $n = 0, 1\ldots$, survive. Thus, we see that positions of the tachyon poles are different if we use the Mandelstam-Sommerfeld–Watson transformation instead of the standard Sommerfeld–Watson transformation. Moreover, the singularities at $\alpha(t) = -(n+1/2)$ do not appear in the amplitude as well, if so-called Mandelstam symmetry of the partial amplitudes with respect to the point $l = -1/2$ is assumed:[9]

$$A^{\xi}_{l-1/2}(t) = A^{\xi}_{-l-1/2}(t). \quad \text{(A.9)}$$

All of the above indicates a fictitious character of the singularities at negative values of $\alpha(t)$.

Appendix B

In this appendix, we will calculate the asymptotics of the right-hand side of (26) at large values of variable $c = ab/R_g(s)$, where b is the impact parameter and a is defined in the text (25). The first quantity under consideration, $I_<(c)$, is represented by the integral

$$I_<(c) = \int_0^1 dz\, z\, J_0(cz)\, e^{-a^2 z^2}. \quad \text{(B.1)}$$

By using the well-known relation between Bessel functions [7],

$$z^{\nu} J_{\nu-1}(cz) = \frac{1}{c}\frac{d}{dz}\big[z^{\nu} J_{\nu}(cz)\big], \quad \text{(B.2)}$$

one can easily obtains from (B.1)

$$\begin{aligned} I_<(c) &= \frac{e^{-a^2}}{c} \sum_{m=0}^{N} \left(\frac{2a^2}{c}\right)^m J_{m+1}(c) \\ &\quad + \left(\frac{2a^2}{c}\right)^{N+1} \int_0^1 dz\, z^{N+2}\, J_{N+1}(cz)\, e^{-a^2 z^2} \\ &= \frac{e^{-a^2}}{c} \sum_{m=0}^{N} \left(\frac{2a^2}{c}\right)^m J_{m+1}(c) + \mathrm{o}(c^{-N-2}) \quad \text{(B.3)} \end{aligned}$$

for any integer $N \geqslant 0$. Thus, we obtain the leading asymptotic terms:

$$I_<(c)\big|_{c\gg 1} = \frac{e^{-a^2}}{c}\left[J_1(c) + \frac{2a^2}{c} J_2(c)\right] + \mathrm{o}(c^{-3}). \quad \text{(B.4)}$$

[9] From the Gribov–Froissart representation, the Mandelstam symmetry follows for all $A^{\xi}_l(t)$ with $l \geqslant N$, where N is a number of subtractions in a dispersion relation for the amplitude, due to the symmetry property of $Q_l(z)$ (A.8). The symmetry takes place in a potential scattering (see [4] for more details).

Note, the sum in (B.3) converges at $N \to \infty$ for any fixed c.

The quantity $I_>(c)$ is represented by the integral

$$I_>(c) = \int_1^{\infty} \frac{dz}{z}\, J_0(cz)\, e^{-a^2 z^2}. \quad \text{(B.5)}$$

In order to estimate $I_>(c)$ at large c, it is convenient to recast it in the form

$$\begin{aligned} &I_>(c) \\ &= \lim_{\alpha\to 0}\left[\int_0^{\infty} dz z^{\alpha-1} J_0(cz) e^{-a^2 z^2} - \int_0^1 dz z^{\alpha-1} J_0(cz) e^{-a^2 z^2}\right] \\ &= \lim_{\alpha\to 0}\left[\int_0^{\infty} dz z^{\alpha-1} J_0(cz)\, e^{-a^2 z^2} - \int_0^1 dz z^{\alpha-1} J_0(cz)\right] \\ &- \sum_{k=1}^{\infty} (-1)^k \frac{a^{2k}}{k!} \int_0^1 dz z^{2k-1} J_0(cz). \quad \text{(B.6)} \end{aligned}$$

By using table integrals from [5], we get (up to power corrections in α):

$$\begin{aligned} &\int_0^{\infty} dz z^{\alpha-1} J_0(cz)\, e^{-a^2 z^2}\bigg|_{\alpha\to 0} \\ &\quad \simeq \frac{a^{-\alpha}}{2}\left[\Gamma\big(\frac{\alpha}{2}\big) - \gamma - \ln\left(\frac{c^2}{4a^2}\right) + \mathrm{Ei}\left(-\frac{c^2}{4a^2}\right)\right], \quad \text{(B.7)} \end{aligned}$$

where γ is the Euler constant and $\mathrm{Ei}(-z)$ is the exponential integral. Analogously, one gets [5]

$$\begin{aligned} \int_0^1 dz z^{\alpha-1} J_0(cz)\bigg|_{\alpha\to 0} &\simeq \frac{1}{2}\left[\Gamma\big(\frac{\alpha}{2}\big) - \gamma - \ln\left(\frac{c^2}{4}\right)\right] \\ &\quad - \int_c^{\infty} \frac{dz}{z}\, J_0(z). \quad \text{(B.8)} \end{aligned}$$

(B.6)–(B.8) result in the expression

$$\begin{aligned} I_>(c) &= \frac{1}{2}\,\mathrm{Ei}\left(-\frac{c^2}{4a^2}\right) + \int_c^{\infty} \frac{dz}{z} J_0(z) \\ &\quad - \sum_{k=1}^{\infty} (-1)^k \frac{a^{2k}}{k!} \int_0^1 dz z^{2k-1} J_0(cz). \quad \text{(B.9)} \end{aligned}$$

By taking into account (B.2), we get (for $N \geqslant 0$)

$$\begin{aligned} \int_c^{\infty} \frac{dz}{z} J_0(z) &= -\frac{1}{c} \sum_{m=0}^{N} m! \left(\frac{2}{c}\right)^m J_{m+1}(c) \\ &\quad + 2^{N+1}(N+1)! \int_c^{\infty} \frac{dz}{z^{N+2}} J_{N+1}(z). \quad \text{(B.10)} \end{aligned}$$

A series, which arises in (B.10) in the limit $N \to \infty$, does not converge. To see this, one should use asymptotics of the Gamma-function [6],

$$\Gamma(m)\Big|_{m\to\infty} \simeq \exp\Big[\Big(m-\frac{1}{2}\Big)\ln m - m + \frac{1}{2}\ln(2\pi)\Big], \quad \text{(B.11)}$$

and an asymptotics of the Bessel function at large value of its index (at fixed c) [8],

$$J_m(c)\Big|_{m\to\infty} \simeq \exp\Big\{m\Big[1+\ln\Big(\frac{c}{2}\Big)\Big] - \Big(m+\frac{1}{2}\Big)\ln m - \frac{1}{2}\ln(2\pi)\Big\}. \quad \text{(B.12)}$$

Then one concludes that the mth term in the series under consideration tends to $(2m)^{-1}$ at $m \to \infty$. So the sum in (B.10) is an asymptotic one at large N.

As a result, we have (for $N \geqslant 0$):

$$\sum_{k=1}^{\infty}(-1)^k \frac{a^{2k}}{k!}\int_0^1 dz z^{2k-1} J_0(cz) = \frac{e^{-a^2}-1}{c} J_1(c) + \sum_{k=1}^{\infty}(-1)^k \frac{a^{2k}}{k}\Big[\frac{1}{c}\sum_{m=1}^{N}(-1)^m \frac{1}{\Gamma(k-m)}\Big(\frac{2}{c}\Big)^m J_{m+1}(c) - (-1)^N\Big(\frac{2}{c}\Big)^{N+1}\frac{1}{\Gamma(k-N-1)}\int_0^1 dz z^{2k-N-2} J_{N+1}(cz)\Big]. \quad \text{(B.13)}$$

Taking into account that $1/\Gamma(-n) = 0$ for any nonnegative integer n, we come to the formula

$$\sum_{k=1}^{\infty}(-1)^k \frac{a^{2k}}{k!}\int_0^1 dz z^{2k-1} J_0(cz) = \frac{e^{-a^2}-1}{c} J_1(c) - \frac{1}{c}\sum_{m=1}^{N}\Big(\frac{2}{c}\Big)^m \gamma(m+1,a^2) J_{m+1}(c) - \Big(\frac{2}{c}\Big)^{N+1}\int_0^1 dz z^{-N-2}\gamma(N+2,z^2a^2) J_{N+1}(cz)\Big]. \quad \text{(B.14)}$$

Here $\gamma(a,x)$ is the incomplete Gamma function. At $N \to \infty$, the sum in the right-hand side of (B.14) converges. It can be easily shown if we use the expansion of the incomplete Gamma function, $\gamma(m+1,1) = \sum_{k=0}^{m}[k!(k+m+1)]^{-1}$, and use (B.12).

It follows from (B.9), (B.10) and (B.14) that[10]

$$I_>(c) = -\frac{e^{-a^2}}{c} J_1(c) - \frac{1}{c}\sum_{m=1}^{N}\Big(\frac{2}{c}\Big)^m m!\, J_{m+1}(c) + \frac{1}{c}\sum_{m=1}^{\infty}\Big(\frac{2}{c}\Big)^m \gamma(m+1,a^2) J_{m+1}(c) + \mathrm{o}(c^{-N-2}), \quad \text{(B.15)}$$

and we obtain the leading asymptotic terms

$$I_>(c)\Big|_{c\gg 1} = -\frac{e^{-a^2}}{c}\Big[J_1(c) + \frac{2(1+a^2)}{c} J_2(c)\Big] + \mathrm{o}(c^{-3}). \quad \text{(B.16)}$$

Notice that the leading terms in (B.4) and (B.16) (proportional to $J_1(c)$) cancel out. Thus, we get

$$[I_<(c) + I_>(c)]\Big|_{c\gg 1} \simeq -\frac{2e^{-a^2}}{c^2} J_2(c), \quad \text{(B.17)}$$

and we arrive at the asymptotics presented in the text (28). The complete asymptotic expansion of the functions $I_<(c)$ and $I_>(c)$ can be obtained if desired from (B.3) and (B.15), respectively.

Appendix C

In order to find $\mathrm{Re}\tilde{A}(s,t)$, we need to calculate the integral

$$M = \int_0^{\infty} db\, b\, J_0(q_\perp b)\Big[I_<\Big(\frac{ab}{R_g(s)}\Big) + I_>\Big(\frac{ab}{R_g(s)}\Big)\Big]. \quad \text{(C.1)}$$

By taking into account formulae from Appendix B, one can get

$$I_<(c) + I_>(c) = -\frac{2e^{-a^2}}{c^2} J_2(c) + \Big(\frac{2a^2}{c}\Big)^2\int_0^1 dz z^3 J_2(cz) e^{-a^2z^2} + \Big(\frac{2}{c}\Big)^2\int_0^1 \frac{dz}{z^3}\gamma(3, z^2a^2) J_2(cz) + 8\int_c^{\infty}\frac{dz}{z^3} J_2(z) + \frac{1}{2}\mathrm{Ei}\Big(-\frac{c^2}{4a^2}\Big), \quad \text{(C.2)}$$

where again we have defined $c = ab/R_g(s)$.

By the use of (B.2) and other relation between the Bessel functions [7],

$$\frac{d}{dz}\big[z^{-\nu} J_\nu(cz)\big] = -cz^{-\nu} J_{\nu+1}(z), \quad \text{(C.3)}$$

the following formula can be derived (α, $\beta > 0$):[11]

$$\int^{x}\frac{dz}{z^\mu} J_\nu(\alpha z) J_{\nu+\mu+1}(\beta z) = \frac{1}{\beta x^\mu}\sum_{m=1}^{\infty}\Big(\frac{\beta}{\alpha}\Big)^m J_{\nu+m}(\alpha x) J_{\nu+\mu+m}(\beta x) + \text{const.} \quad \text{(C.4)}$$

[10] Notice, the exponetial integral $\mathrm{Ei}(-c^2/4a^2)$ decreases exponentially at $c \to \infty$.

[11] A special case of this equation for $\mu = 0$ and $a = b$ can be found in [5].

In particular, one obtains (for A, $q_\perp > 0$)

$$\int\limits_0^\infty \frac{db}{b} J_0(q_\perp b)\, J_2(Ab) = \frac{1}{Ab} \sum_{m=1}^{\infty} \Big(\frac{A}{q_\perp}\Big)^m J_m\big(q_\perp b\big) J_{m+1}(Ab)\Big|_{b=0}^{b=\infty} = 0. \tag{C.5}$$

Thus, the first three terms in the right-hand side of (C.2) gives zero after the integration in variable b in (C.1).

The next to the last term in (C.2) results in

$$8\int\limits_0^\infty db\, b\, J_0(q_\perp b) \int\limits_c^\infty \frac{dz}{z^3}\, J_2(z) = \frac{8R_g(s)}{aq_\perp} \int\limits_0^\infty \frac{dz}{z^2}\, J_2(z)\, J_1\Big[\frac{R_g(s)q_\perp}{a} z\Big]. \tag{C.6}$$

The integral in (C.6) is a table one (see [5]):

$$\int\limits_0^\infty \frac{dz}{z^2} J_2(z) J_1\Big[\frac{R_g(s)q_\perp}{a} z\Big] = \begin{cases} \dfrac{R_g(s)q_\perp}{8a}\Big[2 - \dfrac{R_g^2(s)q_\perp^2}{a^2}\Big], & \text{for} \quad |t| < a^2 R_g^{-2}(s) \\ \dfrac{a}{8R_g(s)q_\perp}, & \text{for} \quad |t| \geqslant a^2 R_g^{-2}(s). \end{cases} \tag{C.7}$$

The contribution from the last term in (C.2) can be also explicitly calculated with the help of a table integral from [5]:

$$\frac{1}{2}\int\limits_0^\infty db\, b\, J_0(q_\perp b)\, \mathrm{Ei}\Big(-\frac{c^2}{4a^2}\Big) = -\frac{1}{q_\perp^2}\, \Big[1 - e^{-q_\perp^2 R_g^2(s)}\Big], \tag{C.8}$$

and we get

$$\begin{aligned} M\Big|_{q_\perp < aR_g^{-1}(s)} &= \frac{R_g^2(s)}{a^2}\Big[2 - \frac{R_g^2(s)q_\perp^2}{a^2}\Big] \\ &\quad - \frac{1}{q_\perp^2}\Big[1 - e^{-q_\perp^2 R_g^2(s)}\Big], \\ M\Big|_{q_\perp \geqslant aR_g^{-1}(s)} &= \frac{1}{q_\perp^2} e^{-q_\perp^2 R_g^2(s)}. \end{aligned} \tag{C.9}$$

As a result, we arrive at the expression for the amplitude presented in the text (35).

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